Incremental Selection of Most-Filtering Conjectures and Proofs of the Selected Conjectures

J. Cheukam Ngouonou[1,2,4], R. Gindullin[1,2], C.-G. Quimper[4], N. Beldiceanu[1,2] and R. Douence[1,2,3]

[1]IMT Atlantique, Nantes, France [2]LS2N, Nantes, France
[3]INRIA, Nantes, France [4]Université Laval, Québec, Canada
ramiz.gindullin@it.uu.se, nicolas.beldiceanu@imt-atlantique.fr, jovial.cheukam-ngouonou.1@ulaval.ca, Remi.Douence@imt-atlantique.fr, Claude-Guy.Quimper@ift.ulaval.ca

Abstract. We present an improved incremental selection algorithm of the selection algorithm presented in [1] and prove all the selected conjectures.

1 Introduction

In Section 2, we describe an incremental algorithm for selecting the most-filtering bound conjectures. This incremental algorithm and the speedup it offers were mentioned in [1], but were not described for space reasons. In Section 3 and in Section 4 we respectively prove the selected conjectures for the Partition and the BinSeq constraints that were not proved in [1].

2 An Incremental Selection Algorithm

We present an incremental version of the selection algorithm described in [1]. Unlike our original algorithms, we do not post the candidate bound constraints from scratch during each step of the dichotomic search; nor do we scan the set of solutions from scratch when looking for the next candidate bound constraint to select.

In the following, we assume that if a constraint fails while being posted, it will be removed by the solver. Given the set of constraints already posted $\mathcal{C}$, the function Labeling($\mathit{FeatVars}, \mathcal{X}$) returns a triplet ($\mathit{NBack}, \mathit{Finished}, \mathit{Sol}$), where Sol is the first solution found that satisfies all the constraints of $\mathcal{C}$ by assigning the variables of $\mathit{FeatVars}$ and $\mathcal{X}$ from left to right, assuming that the variables are fixed by scanning their domains by increasing values:

- *NBack* is the number of backtracks to find a solution or prove that there is no solution,
- *Finished* is set to TRUE if no solution could be found, and to FALSE otherwise,

– *Sol* is meaningless if no solution was found.

The main selection algorithm, Alg. (1), has the following arguments:

– *Ctr* the constraint associated with a combinatorial object, e.g. the PARTITION or the BINSEQ constraints.
– *FeatVars* the set of feature variables of constraint *Ctr*, e.g. variables P, $\underline{M}$, $\overline{M}$, $\overline{\underline{M}}$ and S for PARTITION or variables N_1, G, $\underline{G}$, $\overline{G}$, $\overline{\underline{G}}$, GS, $\underline{D}$, $\overline{D}$, $\overline{\underline{D}}$, DS for BINSEQ.
– $\mathcal{X}$ the array of variables $[X_1, X_2, \ldots, X_n]$ of constraints *Ctr*.
– *Bounds* the list of candidate bound constraints found by the Bound Seeker for constraint *Ctr*.

From a set of candidate bound constraints *Bounds*, Alg. (1) returns a list of selected constraints. Line 1 of Alg. (1) posts the constraint *Ctr* only once during the whole selection process, calls Alg. (2) in line 2 to compute all solutions using all candidate bound constraints, and finally selects in line 3 a subset of candidate bound constraints that lead to the computation of each solution of the constraint *Ctr* without increasing the number of backtracks.

Algorithm 1: Selection(*Ctr*, *FeatVars*, $\mathcal{X}$, *Bounds*)

```
1 post constraint Ctr(FeatVars, X);
2 Sols ← ComputeAllSolutions(FeatVars, X, Bounds);
3 return Select(Sols, FeatVars, X, Bounds, {});
```

Alg. (2) and (3) compute all solutions of the feature variables *FeatVars* wrt constraint *Ctr* in ascending lexicographic order on the *FeatVars* variables and records each solution with the number of backtracks to find it. Since we will need to calculate a specific solution later on in the selection process, independently from the other solutions, we proceed as follows: to obtain the i-th solution wrt the lexicographic order of *FeatVars*, we compute the smallest lexicographic solution that is strictly greater than the $(i-1)$-th solution already known. In lines 3–5 of Alg. (3) we post the constraint $\mathit{FeatVars} >_{\ell ex} \mathit{Sol}$ stating that *FeatVars* is lexicographically strictly greater than *Sol* before computing the next smallest lexicographic solution. Finally, in line 3 of Alg. (2), we sort all solutions by increasing number of backtracks, as the selection process will use this order to reduce the time spent generating solutions.

Alg. 4 is a recursive selection algorithm that selects a subset of bound constraints that does not increase the number of backtracks to find each solution. At each step of the recursion, Alg. 4 successively:

– Post the previously selected bound constraint *PrevBound* (lines 1–2), as all previously selected bound constraints must be posted when searching for the next bound constraint to select. Note that unlike the selection algorithm described in [1], each selected bound constraint is only posted once during the entire selection process.

Algorithm 2: ComputeAllSolutions($\mathit{FeatVars}, \mathcal{X}, \mathit{Bounds}$)

$\forall\, \mathit{Bound} \in \mathit{Bounds}$: post bound constraint *Bound* on *FeatVars*;
$\mathit{Sols} \leftarrow$ EnumerateAllSolutions($\mathit{FeatVars}, \mathcal{X}$);
$\mathit{SortedSols} \leftarrow$ sort *Sols* by increasing number of backstracks;
remove all posted bound constraints;
return *SortedSols*;

Algorithm 3: EnumerateAllSolutions($\mathit{FeatVars}, \mathcal{X}$)

$\mathit{ISol} \leftarrow 0$; $\mathit{Sols} \leftarrow \emptyset$;
while TRUE **do**
    if $\mathit{ISol} > 0 \;\wedge\;$ *post constraint FeatVars* $>_{\ell ex}$ *Sol fails* **then**
        return $\mathit{Sols} \cup \{(\mathit{ISol}, 0, [])\}$;
    $(\mathit{NBack}, \mathit{Finished}, \mathit{Sol}) \leftarrow$ Labeling($\mathit{FeatVars}, \mathcal{X}$);
    $\mathit{Sols} \leftarrow \mathit{Sols} \cup \{(\mathit{ISol}, \mathit{NBack}, \mathit{Sol})\}$;
    if $\mathit{ISol} > 0$ **then** remove lexicographic constraint that was posted;
    if *Finished* **then return** *Sols* **else** $\mathit{ISol} \leftarrow \mathit{ISol} + 1$;

- Select the next bound constraint $\mathit{Selected}'$ from the current set of candidate bound constraints *Bounds*, and create the new reduced set of candidate bound constraints Bounds' (line 3).
- If both, we could select a bound constraint and we still have some candidate bound constraints (line 4), we recursively call Alg. 4 to select the next bound constraints to keep (line 5).

Algorithm 4: Select($\mathit{Sols}, \mathit{FeatVars}, \mathcal{X}, \mathit{Bounds}, \mathit{PrevBound}$)

1 **if** $\mathit{PrevBound} \neq \emptyset$ **then**
2   post previous selected bound constraint *PrevBound* on *FeatVars*;
3 $(\mathit{Selected}', \mathit{Bounds}') \leftarrow$ SelectOne(TRUE,Sols,Sols,$\mathit{FeatVars}$,$\mathcal{X}$,Bounds);
4 **if** $\mathit{Selected}' \neq \emptyset \wedge \mathit{Bounds}' \neq \emptyset$ **then**
5     $\mathit{RestSelected} \leftarrow$ Select($\mathit{Sols}, \mathit{FeatVars}, \mathcal{X}, \mathit{Bounds}', \mathit{Selected}'$);
6   **return** $\mathit{Selected}' \cup \mathit{RestSelected}$;
7 **else** **return** $\mathit{Selected}'$;

Alg. 5 recursively selects the next bound constraint from the list of candidate bound constraints *Bounds*. At each stage of the selection process, we incrementally post a suffix of the list of candidate bound constraints *Bounds*, i.e. each candidate bound constraint is only posted at most once during the search for the next candidate.

- Lines 1–5 split the list of candidate bound constraints *Bounds* in a prefix and suffix part in an uneven way where the suffix is smaller than the prefix part. The current way to split *Bounds* was determined experimentally by

testing different manners of partitioning on different examples. We prefer to incrementally add a limited number of bound constraints so that we end up in a situation where we do not have enough constraints and this increases the number of backtracks needed to find a solution. Otherwise, adding too many bound constraints would result in not increasing the number of backtracks, which would have the effect of scanning all remaining solutions in *Sols* and deleting the added constraints in order to post a smaller set of constraints. As a result, the same solution from *Sols* would be generated several times, which can be mitigated by limiting the number of constraints we added.
- Line 6 performs a dichotomic search on the suffix and prefix parts of the candidate list to return a selected candidate $Selected'$ and the remaining list of candidate constraints $Bounds'$.
- When we are at the top-level call of SelectOne, i.e. when SelectOne is called in line 3 of Alg. (4), lines 7–8 of Alg. (5) remove any bound constraints posted within the call to Alg. (6) on line 6 of Alg. (4) to prepare for the next call to SelectOne from Select.

Algorithm 5: SelectOne($Top, Sols, AllSols, FeatVars, \mathcal{X}, Bounds$)

1 $Len \leftarrow |Bounds|$;
2 **if** $Len > 200$ **then** $Mid \leftarrow Len - 100$ **else if** $Len < 3$ **then** $Mid \leftarrow \lfloor \frac{Len+1}{2} \rfloor$
3 **else** $Mid \leftarrow \lfloor \frac{2 \cdot Len+2}{3} \rfloor$;
4 $Prefix \leftarrow$ first Mid elements of $Bounds$;
5 $Suffix \leftarrow$ last $Len - Mid$ elements of $Bounds$;
6 $(Selected', Bounds') \leftarrow$ Dicho($Sols, AllSols, FeatVars, \mathcal{X}, Len, Prefix, Suffix$);
7 **if** Top **then**
8 remove all bound constraints posted from the current call to Dicho;
9 **return** $(Selected', Bounds')$;

Alg. (6) performs a dichotomic search wrt the suffix and prefix parts of the candidate list.

- Line 1 posts all candidate bound constraints from *Suffix*.
- For each solution *Sols*, line 2 computes the number of backtracks to obtain the next solution, and stops when the number of backtracks increases i.e. $MissingBound$ = TRUE, or when the list of solutions is fully explored i.e. $MissingBound$ = FALSE. The set $Sols'$ corresponds to the set $Sols'$ from which we removed all the solutions leading to the same number of backtracks.
- 1. If the number of backtracks increases and we can still add several bound constraints to the constraints to keep (see line 3), we continue the dichotomic search by using the bound constraints of the *Prefix* to select the next bound to keep (see lines 4–5).
  2. If we can only add one bound constraint (see line 7), we return that bound constraint if the number of backtracks increases (see **then** part in line 8); otherwise we do not select a bound constraint (see **else** part in line 8).

3. Otherwise, since adding all bound constraints of *Suffix* (see line 1) was sufficient to avoid increasing the number of backtracks when searching for all solutions, we look for the next candidate bound constraint to select in the suffix (see line 10).

Algorithm 6: Dicho(*Sols*, *AllSols*, *FeatVars*, $\mathcal{X}$, *Len*, *Prefix*, *Suffix*)

```
1  ∀ Bound ∈ Suffix : post bound constraint Bound on FeatVars;
2  (Sols′, MissingBound) ← Enumerate(Sols, AllSols, FeatVars, 𝒳);
3  if MissingBound ∧ Len > 1 then
4      (Selected′,Bounds)←SelectOne(FALSE,Sols′, AllSols,FeatVars,𝒳,Prefix);
5      return (Selected′, Bounds ∪ Suffix);
6  remove all bound constraints posted on line 1;
7  if Len = 1 then
8      if MissingBound then return(Prefix, ∅) else return(∅, Prefix)
9  else
10     return SelectOne(FALSE, Sols, AllSols, FeatVars, 𝒳, Suffix)
```

Alg. 7 iteratively computes the next solution of each solution in *Sols*, considering the current set of posted bound constraints, until backtracking increases wrt the number of backtracks obtained using all bound constraints, or until no more solutions exist. The *next solution* of the *ISol*-th solution *Sol* of the set *Sols* is the smallest lexicographic solution strictly greater than *Sol* (see lines 1–4). If the current solution *Sol* does not increase the number of backtracks (line 7), it is removed from the list of solutions *Sols* to be checked (line 8), and the algorithm continues to check the remaining solutions *Sols′* (line 10); otherwise, if backtracking increases, the check is terminated (line 12).

Algorithm 7: Enumerate(*Sols*, *AllSols*, *FeatVars*, $\mathcal{X}$)

```
1  let (ISol, −, Sol) be the first element of Sols;  JSol ← ISol + 1;
2  let (JSol, NBack, −) be the JSol-th element of AllSols;  Finished ← FALSE;
3  if ISol > 0 ∧  post constraint FeatVars >ℓex Sol fails then
4      NBack ← 0; Finished ← TRUE;
5  if ¬Finished then (Back, Finished, −) ← Labeling(FeatVars, 𝒳);
6  if ISol > 0 then remove lexicographic constraint that was posted;
7  if Back = NBack then
8      Sols′ ← Sols − {(I, Bi, Si)};
9      if Sols′ ≠ ∅ then
10         return Enumerate(Sols′, AllSols, FeatVars, 𝒳);
11     else  return (Sols′, FALSE) ;
12 else  return (Sols, TRUE) ;
```

3 Proofs for the Conjectures of the PARTITION Constraint

We borrow the definition of the PARTITION constraint from [1].

Definition 1. PARTITION$([X_1, X_2, \dots, X_n], P, \underline{M}, \overline{M}, \overline{\underline{M}}, S)$ *is satisfied iff*

$$P = |\{X_1, X_2, \dots, X_n\}| \qquad S = \sum_{j \in \mathcal{X}} |\{i \mid X_i = j\}|^2 \tag{1}$$

$$\underline{M} = \min_{j \in \mathcal{X}} |\{i \mid X_i = j\}| \qquad \overline{M} = \max_{j \in \mathcal{X}} |\{i \mid X_i = j\}| \qquad \overline{\underline{M}} = \overline{M} - \underline{M} \tag{2}$$

We prove the following three selected conjectures, which were found by the Bound Seeker.

– An upper bound on S and two distinct upper bounds on $\overline{\underline{M}}$:

$$S \leq MID^2 + SM \cdot RR + SMIN \tag{3}$$

with:

$$MID = \begin{cases} \underline{M} + (R \text{ **mod** } \overline{\underline{M}}) & \text{if } \overline{\underline{M}} > 0 \\ \underline{M} & \text{otherwise} \end{cases} \tag{4}$$

$$R = n - P \cdot \underline{M} \tag{5}$$

$$RR = \begin{cases} \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor & \text{if } \overline{\underline{M}} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$SM = \overline{M}^2 - \underline{M}^2 \tag{7}$$

$$SMIN = \underline{M}^2 \cdot (P - 1) \tag{8}$$

– Two distinct upper bounds on $\overline{\underline{M}}$:

$$\overline{\underline{M}} \leq n - P \cdot \underline{M} \tag{9}$$

$$\overline{\underline{M}} \leq \min(P \cdot \overline{M} - n, \overline{M} - 1) \tag{10}$$

We provide the proofs of correctness for three selected bounds (3), (9), and (10).

To prove the conjecture (3), we first prove the five following lemmas.

Lemma 1. *If there exist at least two partitions whose sizes are strictly between* $\underline{M}$ *and* $\overline{M}$*, in other words, if their sizes are* $\underline{M} + r_1$ *and* $\underline{M} + r_2$ *such that* $1 \leq r_1 \leq r_2 < \overline{\underline{M}}$*, then* S *is not maximal.*

Proof. We have

$$\begin{aligned} (\underline{M} + r_1 - 1)^2 + (\underline{M} + r_2 + 1)^2 = (\underline{M} + r_1)^2 + \\ (\underline{M} + r_2)^2 + 2(r_2 - r_1) + 2 > (\underline{M} + r_1)^2 + \\ (\underline{M} + r_2)^2 \end{aligned} \tag{11}$$

Let O_i (with $i \in [1 : P]$) be the sizes of the P partitions of n elements. It means that we have $n = \sum_i^P O_i$ and $S = \sum_i^P O_i^2$. The two terms of (11) appear in the computation of S. In other words, it is possible to remove one element from partition 1 and add it to partition 2 to obtain a larger sum of squares without affecting the other terms in the computation of S, since the value of $\sum_i^P O_i = n$ remains unchanged. This proves S is not maximal. □

Lemma 2. *Let o_{min} and o_{max} be the number of partitions that have respectively the size of $\underline{M}$ and $\overline{M}$. If $(R \bmod \overline{M} > 0)$, then there is at least one partition whose size is strictly between $\underline{M}$ and $\overline{M}$.*

Proof. By contradiction, suppose that $R \bmod \overline{M} > 0$ and all partitions are either of size $\underline{M}$ or $\overline{M}$. By definition of R we have:

$$R = \sum_{i=1}^{P} (O_i - \underline{M}) = \sum_{i=1}^{o_{max}} (\overline{M} - \underline{M}) + \sum_{i=1}^{o_{min}} (\underline{M} - \underline{M}) = o_{max} \cdot \overline{M} \tag{12}$$

which contradicts that $R \bmod \overline{M} > 0$. So $R \bmod \overline{M} > 0$ implies that there exists a partition whose size is strictly between $\underline{M}$ and $\overline{M}$. □

Lemma 3. *Let o_{min} and o_{max} be the number of partitions that have respectively the size of $\underline{M}$ and $\overline{M}$. If S is maximal and $\overline{M} > 0 \wedge (R \bmod \overline{M} > 0)$, then only one partition whose size is strictly between $\underline{M}$ and $\overline{M}$ exists, and its size is equal to $\underline{M} + R \bmod \overline{M}$.*

Proof. According to Lemma 1, as S is maximal, there is at most one partition whose size is strictly between $\underline{M}$ and $\overline{M}$ and according to Lemma 2, as $R \bmod \overline{M} > 0$ there is at least one partition whose size is strictly between $\underline{M}$ and $\overline{M}$. So there is only one partition whose size is strictly between $\underline{M}$ and $\overline{M}$. So let O_P be the size of that partition and $O_i, \forall i \in [1 : P - 1]$ the sizes of the remaining partitions. Let also $r^* = O_P - \underline{M}$.

Then we have $0 < r^* < \overline{M}$ because $\underline{M} < O_P < \overline{M}$. So, by definition, we have

$$R = \sum_{i=1}^{P} (O_i - \underline{M}) = \sum_{i=1}^{o_{max}} (\overline{M} - \underline{M}) + \sum_{i=o_{max}+1}^{o_{max}+o_{min}} (\underline{M} - \underline{M}) + r^* = o_{max} \cdot \overline{M} + r^* \text{ with } r^* < \overline{M} \tag{13}$$

According to the definition of Euclidean division, the relation (13) is equivalent to $r^* = R \bmod \overline{M}$. So $O_P = \underline{M} + R \bmod \overline{M}$. □

Lemma 4. *Let o_{min} and o_{max} be the number of partitions that have respectively the size of $\underline{M}$ and $\overline{M}$. If S is maximal and $\overline{\underline{M}} > 0 \wedge (R \bmod \overline{\underline{M}} = 0)$, then there is no partition whose size is strictly between $\underline{M}$ and $\overline{M}$.*

Proof. By contradiction, suppose that S is maximal, $\overline{\underline{M}} > 0 \wedge (R \bmod \overline{\underline{M}} = 0)$ and there are k ($k \geq 1$) partitions whose size is strictly between $\underline{M}$ and $\overline{M}$ and are denoted by $I_i, \forall i \in [1 : k]$. Then, by definition

$$R = \sum_{i=1}^{o_{max}} (\overline{M} - \underline{M}) + \sum_{i=1}^{k} (I_i - \underline{M}) = o_{max} \cdot \overline{\underline{M}} + \sum_{i=1}^{k} (I_i - \underline{M}) \tag{14}$$

$$\text{with } I_i - \underline{M} < \overline{\underline{M}}, \forall i \in [1 : k] \tag{15}$$

Because $R \bmod \overline{\underline{M}} = 0$, we have $R = \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor \cdot \overline{\underline{M}}$. So according to (14), $\forall i \in [1 : k]$ with $I_i - \underline{M} < \overline{\underline{M}}$, we have

$$o_{max} \cdot \overline{\underline{M}} + \sum_{i=1}^{k} (I_i - \underline{M}) = \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor \cdot \overline{\underline{M}} \tag{16}$$

So according to (16), $\overline{\underline{M}}$ is a divisor of $\sum_{i=1}^{k} (I_i - \underline{M})$. Which implies that $k \geq 2$, because of (15). And according to Lemma 1, $k \geq 2$ implies that S is not maximal. Which is a contradiction. □

Lemma 5. *Let o_{min} and o_{max} be the number of partitions that have respectively the size of $\underline{M}$ and $\overline{M}$.*

Then $\left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor$ (resp. $P - \left\lceil \frac{R}{\overline{\underline{M}}} \right\rceil$) is a tight upper bound of o_{max} (resp. o_{min}).

Proof. We upper bound the values of o_{max} by using the definition of n. We have:

$$n = \overline{M} \cdot o_{max} + \sum_{i=1}^{P - o_{max}} o_i \geq \overline{M} \cdot o_{max} +$$

$$(P - o_{max}) \cdot \underline{M} \geq (\overline{M} - \underline{M}) \cdot o_{max} + P \cdot \underline{M} \tag{17}$$

$$\Longrightarrow o_{max} \leq \frac{n - P \cdot \underline{M}}{\overline{M} - \underline{M}} \tag{18}$$

And by definition, $R = n - P \cdot \underline{M}$ and $\overline{\underline{M}} = \overline{M} - \underline{M}$. So:

$$o_{max} \leq \frac{R}{\overline{\underline{M}}} \tag{19}$$

Symmetrically, we upper bound o_{min}:

$$n = \underline{M} \cdot o_{min} + \sum_{i=1}^{P-o_{min}} o_i \leq \underline{M} \cdot o_{min} +$$

$$\overline{M} \cdot (P - o_{min}) = -(\overline{M} - \underline{M}) \cdot o_{min} + \overline{M} \cdot P \tag{20}$$

$$\Longrightarrow o_{min} \leq \frac{\overline{M} \cdot P - n}{\overline{M} - \underline{M}} = \frac{(\overline{\underline{M}} + \underline{M}) \cdot P - n}{\overline{\underline{M}}} =$$

$$P + \frac{-(n - \underline{M} \cdot P)}{\overline{\underline{M}}} = P - \frac{R}{\overline{\underline{M}}} \tag{21}$$

Since o_{max}, $o_{min} \in \mathbb{N}$, we have

$$o_{max} \leq \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor \qquad \text{and} \qquad o_{min} \leq P - \left\lceil \frac{R}{\overline{\underline{M}}} \right\rceil \tag{22}$$

Finally, according to Lemmas 3 and 4, when S is maximal, we have $R = o_{max} \cdot \overline{\underline{M}} + R \bmod \overline{\underline{M}}$ and at most one partition exists whose size is strictly between $\underline{M}$ and $\overline{M}$. Which means that if S is maximal, we have:

$$o_{max} = \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor \tag{23}$$

$$\Longrightarrow o_{min} = P - o_{max} - \alpha = P - \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor - \alpha \tag{24}$$

$$\text{with } \alpha = 0 \text{ if } R \bmod \overline{\underline{M}} = 0 \text{ and } \alpha = 1 \text{ if not.} \tag{25}$$

$$\Longrightarrow o_{min} = P - \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor - \alpha = P - \left\lceil \frac{R}{\overline{\underline{M}}} \right\rceil \tag{26}$$

□

3.1 Conjecture (3)

Proof (Conjecture (3)*).*

- When $\overline{\underline{M}} = \overline{M} - \underline{M} = 0$. The sizes of the partitions are all the same. Then, according to equations (5) to (8) we have $MID = \underline{M}, SM = RR = 0, SMIN = \underline{M}^2 \cdot (P-1)$. By substituting these MID, SM, RR and $SMIN$ in (3), we obtain

$$S \leq \underline{M}^2 + 0 + \underline{M}^2 \cdot (P-1) = \underline{M}^2 \cdot P \tag{27}$$

  which is consistent with the definition of S because, as the sizes of the P partitions are all the same, they are equal to $\underline{M}$. Which means that $S = \underline{M}^2 \cdot P$.

- When $\overline{\underline{M}} > 0 \wedge (R \bmod \overline{\underline{M}} > 0)$. Let o_{min} and o_{max} be the number of partitions that have respectively the size of $\underline{M}$ and $\overline{M}$.
  According to Lemmas 3 and 5, the maximal value S^* of S is

$$S^* = \overline{M}^2 \cdot o_{max} + \underline{M}^2 \cdot o_{min} + (\underline{M} + R \bmod \overline{\underline{M}})^2 \quad (28)$$
$$\text{with } o_{max} = \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor \text{ and } o_{min} = P - \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor - 1$$

  So because $R \bmod \overline{\underline{M}} > 0$, we have according to equations (5) to (8), $MID = \underline{M} + R \bmod \overline{\underline{M}}, SM = \overline{M}^2 - \underline{M}^2, RR = \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor, SMIN = \underline{M}^2 \cdot (P-1)$.
  By substituting these MID, SM, RR and $SMIN$ in (3), we obtain $S \leq S^*$ according to (28), which is consistent.
- When $\overline{\underline{M}} > 0 \wedge (R \bmod \overline{\underline{M}} = 0)$. Let o_{min} and o_{max} be the number of partitions that have respectively the size of $\underline{M}$ and $\overline{M}$.
  According to Lemmas 4 and 5, the maximal value S^* of S is

$$S^* = \overline{M}^2 \cdot o_{max} + \underline{M}^2 \cdot o_{min} \quad (29)$$
$$\text{with } o_{max} = \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor \text{ and } o_{min} = P - \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor$$

  So because $R \bmod \overline{\underline{M}} = 0$, we have according to equations (5) to (8), $MID = \underline{M}, SM = \overline{M}^2 - \underline{M}^2, RR = \left\lfloor \frac{R}{\overline{\underline{M}}} \right\rfloor, SMIN = \underline{M}^2 \cdot (P-1)$. By substituting these MID, SM, RR and $SMIN$ in (3), we obtain $S \leq S^*$ according to (29), which is consistent. □

3.2 Conjecture (9)

Proof (Conjecture (9)).

- If $P = 1$, then $\underline{M} = \overline{M} = n$. Thus $\overline{\underline{M}} = 0$ and $n - P \cdot \underline{M} = n - \underline{M} = 0$. Which implies that $\overline{\underline{M}} \leq n - P \cdot \underline{M}$.
- If $P \geq 2$, then the number of values is equal to the size of the largest partition plus the size of the smallest partition plus the size of all the other partitions.

$$n = \overline{M} + \underline{M} + \sum_{i=1}^{P-2} O_i \quad (30)$$

  As $\underline{M}$ is the size of the smallest partition, we have $O_i \geq \underline{M}$.

$$n \geq \overline{M} + \underline{M} + \sum_{i=1}^{P-2} \underline{M} = \overline{M} + (P-1) \cdot \underline{M} \quad (31)$$

$$n - P \cdot \underline{M} \geq \overline{M} - \underline{M} \quad (32)$$

  Using the definition $\overline{\underline{M}} = \overline{M} - \underline{M}$ we obtain $n - P \cdot \underline{M} \geq \overline{\underline{M}}$.

– Tightness of the conjecture (9): We can construct for every possible value of n, P and $\underline{M}$ the set of partitions so that $n - P \cdot \underline{M} = \overline{M}$, by setting only one of the partitions to size $\overline{M} = n - (P-1) \cdot \underline{M}$ and the rest to size $\underline{M}$. Because, in that case, we have:

$$n = \overline{M} + (P-1) \cdot \underline{M} \tag{33}$$

$$n - P \cdot \underline{M} = \overline{M} + (P-1) \cdot \underline{M} - P \cdot \underline{M} = \overline{M} - \underline{M} = \overline{M} \tag{34}$$

□

3.3 Conjecture (10)

Proof (Conjecture (10)*).*

– If $P = 1$, then $\underline{M} = \overline{M} = n$. Thus $\underline{\overline{M}} = 0$ and $P \cdot \overline{M} - n = \overline{M} - n = 0$. Which implies that $\underline{\overline{M}} \leq P \cdot \overline{M} - n$. And because $\overline{M} \geq 1$, we also have $\underline{\overline{M}} = 0$ and $0 \leq \overline{M} - 1$. Which implies that $\underline{\overline{M}} \leq \overline{M} - 1$. Since two quantities bound $\underline{\overline{M}}$, the smallest of them bounds $\underline{\overline{M}}$. Hence $\underline{\overline{M}} \leq \min(P \cdot \overline{M} - n, \overline{M} - 1)$.
– If $P \geq 2$, then we first show $\underline{\overline{M}} \leq P \cdot \overline{M} - n$.

$$n = \overline{M} + \underline{M} + \sum_{i=1}^{P-2} O_i \tag{35}$$

$$n \leq \overline{M} + \underline{M} + \sum_{i=1}^{P-2} \overline{M} = (P-1)\overline{M} + \underline{M} \tag{36}$$

$$\overline{M} - \underline{M} \leq P \cdot \overline{M} - n \tag{37}$$

$$\underline{\overline{M}} \leq P \cdot \overline{M} - n \tag{38}$$

The largest range one can obtain is when one element is alone in a partition and the remaining $n-1$ elements are together in the 2nd partition. We have $\underline{\overline{M}} \leq \overline{M} - 1$. Since $\underline{\overline{M}}$ is bounded by two quantities, it is bounded by the smallest one, hence $\underline{\overline{M}} \leq \min(P \cdot \overline{M} - n, \overline{M} - 1)$.
– Tightness of the conjecture (10):
For the case $P = 1$, we have $\underline{M} = \overline{M} = n$. Thus $\underline{\overline{M}} = 0$ and $P \cdot \overline{M} - n = \overline{M} - n = 0$. Which implies that $\underline{\overline{M}} = P \cdot \overline{M} - n = 0$. So we have $0 = \min(P \cdot \overline{M} - n, \overline{M} - 1)$. Which implies that $\underline{\overline{M}} = \min(P \cdot \overline{M} - n, \overline{M} - 1)$. So, the bound is tight.
For the case of $P \geq 2$, we can construct for every possible value of n, P and $\overline{M}$ the set of partitions with $P \cdot \overline{M} - n = \underline{\overline{M}}$ or $\overline{M} - 1 = \underline{\overline{M}}$, either by setting only one of the partitions to size $\underline{M} = n - (P-1) \cdot \overline{M}$ and the rest to size $\overline{M}$ if $n > (P-1) \cdot \overline{M}$ or either by setting one of the partitions to size 1 if $n \leq (P-1) \cdot \overline{M}$. Because we have:

- If $n > (P-1) \cdot \overline{M}$, then $n = \underline{M} + (P-1) \cdot \overline{M}$. Which implies that $P \cdot \overline{M} - n = P \cdot \overline{M} - \underline{M} - (P-1) \cdot \overline{M}$. So $P \cdot \overline{M} - n = \overline{M} - \underline{M} = \underline{\overline{M}}$.

- If $n \leq (P-1)\cdot\overline{M}$, we have $P > 2$ because if $P = 2$, it implies that $n \leq \overline{M}$. This implies that $n = \overline{M}$, meaning that $P = 1$, which is inconsistent with $P = 2$. So to reach the tightness, we set one partition to size $\overline{M}$ and another to size 1. And because $P > 2$ in this case, we can set the remaining $P-2$ partitions to size $\left\lfloor \frac{n-\overline{M}-1}{P-2} \right\rfloor$ and size $\left\lceil \frac{n-\overline{M}-1}{P-2} \right\rceil$. Indeed, we have $1 \leq \left\lfloor \frac{n-\overline{M}-1}{P-2} \right\rfloor < \overline{M}$. Because first, $n-\overline{M}-1$ is the remaining number of elements to partition into $P-2$ non-empty sets. So $P-2 \leq n-\overline{M}-1$, which leads to $1 \leq \left\lfloor \frac{n-\overline{M}-1}{P-2} \right\rfloor$. And second, we also have $n \leq (P-1)\cdot\overline{M}$ equivalent to

$$n - \overline{M} - 1 \leq (P-1)\cdot\overline{M} - \overline{M} - 1 \tag{39}$$

$$\frac{n-\overline{M}-1}{P-2} \leq \frac{(P-2)\cdot\overline{M}-1}{P-2} \leq \overline{M} - \frac{1}{P-2} < \overline{M} \tag{40}$$

$$\Rightarrow \left\lfloor \frac{n-\overline{M}-1}{P-2} \right\rfloor < \overline{M} \tag{41}$$

$\square$

4 Proofs for the Conjectures of the BinSeq Constraint

We borrow the definition of the BinSeq constraint from [1].

Definition 2. *The* BinSeq$([X_1, X_2, \ldots, X_n], N_1, G, \underline{G}, \overline{G}, \underline{\overline{G}}, GS, \underline{D}, \overline{D}, \underline{\overline{D}}, DS)$ *constraint is satisfied iff*

- $X_1, X_2, \ldots, X_n$ *is a sequence of 0/1,*
- N_1 *is the number of values* 1 *in the sequence,*
- G *is the number of stretches of 1s,*
- $\underline{G}$ *(resp.* $\overline{G}$*) is the length of the smallest (resp. longest) stretch of 1s,*
- $\underline{\overline{G}}$ *is the difference between the lengths of the longest and the smallest stretch,*
- GS *is the sum of the squared lengths of the stretches of 1s,*
- $\underline{D}$ *(resp.* $\overline{D}$*) is the length of the smallest (resp. longest) inter-distance of 0s,*
- $\underline{\overline{D}}$ *is the difference* $\overline{D} - \underline{D}$*,*
- DS *is the sum of the squared lengths of the inter-distances of 0s.*

When there is no stretch, $\underline{G} = \overline{G} = 0$*; when there is no inter-distance,* $\underline{D} = \overline{D} = 0$.

4.1 Conjecture (42)

We prove the selected conjecture (42), which was found by the Bound Seeker.

$$N_1 \leq \min(G \cdot \overline{G}, n - G + 1) \tag{42}$$

Proof (Conjecture (42)*).* Let g_i (with $i \in [1:G]$) be the number of 1 in the i-th stretch of 1s. We have

$$N_1 = \sum_{i=1}^{G} g_i \leq G \cdot \overline{G} \tag{43}$$

If $G = 0$ then no stretch of 1s appears in the binary sequence. Which means that $N_1 = 0 = \min(0, n - G + 1) = \min(G \cdot \overline{G}, n - G + 1)$.

If $G \geq 1$ then there are $G - 1$ inter-distances of 0s of lengths at least equal to 1. This means that $N_1 \leq n - (G - 1)$. Also, thanks to (43) we have $N_1 \leq G \cdot \overline{G}$. So $N_1 \leq \min(G \cdot \overline{G}, n - G + 1)$. □

4.2 Conjecture (44)

We prove the selected conjecture (44), which was found by the Bound Seeker.

$$\overline{G} \geq \left\lfloor \frac{n}{n - N_1 + 1} \right\rfloor \tag{44}$$

Proof (Conjecture (44)*).* Because G is the number of stretches of 1s, then $G - 1$ is the number of inter-distances of 0s. So the minimum number of 0s in the binary sequence is $G - 1$. Therefore we have

$$n \geq N_1 + G - 1 \Longleftrightarrow n - N_1 + 1 \geq G \tag{45}$$

If $\overline{G} > 0$, thanks to (43), we have $G \geq \frac{N_1}{\overline{G}}$ which leads to

$$n - N_1 + 1 \geq G \geq \frac{N_1}{\overline{G}} \Longrightarrow \overline{G} \geq \frac{N_1}{n - N_1 + 1} \tag{46}$$

And because $\overline{G}$ is an integer, we have $\overline{G} \geq \left\lceil \frac{N_1}{n - N_1 + 1} \right\rceil$. According to [2] for the positive integers n and m with $m \in \mathbb{N}^*$, we have

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n + m - 1}{m} \right\rfloor \tag{47}$$

So according to (47)

$$\overline{G} \geq \left\lfloor \frac{N_1 + n - N_1 + 1 - 1}{n - N_1 + 1} \right\rfloor = \left\lfloor \frac{n}{n - N_1 + 1} \right\rfloor \tag{48}$$

If $\overline{G} = 0$, then $N_1 = 0$. So we have

$$\overline{G} = 0 = \left\lfloor \frac{n}{n + 1} \right\rfloor = \left\lfloor \frac{n}{n - N_1 + 1} \right\rfloor \tag{49}$$

□

4.3 Conjecture (50)

We prove the selected conjecture (50), which was found by the Bound Seeker.

$$\overline{G} \leq \begin{cases} n + \underline{G} & \text{if } \underline{G} = n \cdot \overline{D} \\ \left\lfloor \dfrac{n - \overline{G} - \overline{D} - \min(\underline{D}, 1) - 1}{\min(\underline{D}, 1) + 2} \right\rfloor + \underline{G} & \text{otherwise} \end{cases} \tag{50}$$

Proof (Conjecture (50)*).* If $\underline{G} = n \cdot \overline{D}$, by definition $\overline{G} \leq n$. And as $\underline{G} \geq 0$, we have $\overline{G} \leq n + \underline{G}$.

Otherwise, if $\underline{G} \neq n \cdot \overline{D}$, we consider the case when $\overline{D} = 0$ and $\underline{G} \geq 1$ as well as the case when $\overline{D} \geq 1$:

- In the case $\overline{D} = 0$ and $\overline{G} \geq 1$, we have $\min(\underline{D}, 1) = 0$ and $\overline{G} > \underline{G}$. This means that there are at least two stretches of 1 and an inter-distance of 0s between them in the binary sequence. So the binary sequence has at least two stretches of lengths $\underline{G}$ and $\overline{G}$ because $\underline{G} \geq 1$. This also means that it has an inter-distance of length at least equal to 1, which means that $\underline{G} \leq n - \overline{G} - 1$. Then we have

$$\underline{G} \leq n - \overline{G} - 1 \iff 2 \cdot \underline{G} \leq n - \overline{G} + \underline{G} - 1 \tag{51}$$
$$\iff 2 \cdot \underline{G} \leq n - \underline{G} - 1 \tag{52}$$
$$\implies \underline{G} \leq \left\lfloor \frac{n - \underline{G} - 1}{2} \right\rfloor \tag{53}$$
$$\implies \overline{G} \leq \left\lfloor \frac{n - \underline{G} - 1}{2} \right\rfloor + \underline{G} \tag{54}$$
$$\iff \overline{G} \leq \left\lfloor \frac{n - \overline{G} - \overline{D} - \min(\underline{D}, 1) - 1}{\min(\underline{D}, 1) + 2} \right\rfloor + \underline{G} \tag{55}$$

- In the case $\overline{D} \geq 1$, the binary sequence has at least two inter-distances of 0s. This means that there are also at least three stretches of 1s. So we have

$$n - \overline{D} - \underline{D} - 2 \cdot \underline{G} - \overline{G} \geq 0 \tag{56}$$
$$n - (\overline{D} + \underline{D}) - \underline{D} - 2 \cdot \underline{G} - \overline{G} \geq 0 \tag{57}$$
$$n - \overline{D} - 2 \cdot \underline{D} - 2 \cdot \underline{G} - \overline{G} \geq 0 \tag{58}$$
$$2 \cdot \underline{G} \leq n - \overline{D} - \overline{G} - 2 \cdot \underline{D} \tag{59}$$

As $\underline{D} \geq 1$, we have $-2 \cdot \underline{D} \leq -2$. So we have

$$2 \cdot \underline{G} \leq n - \overline{D} - \overline{G} - 2 \cdot \underline{D} \leq n - \overline{D} - \overline{G} - 2 \tag{60}$$

This leads to $2 \cdot \underline{G} \leq n - \overline{D} - \overline{G} - 2$, which leads to

$$3 \cdot \underline{G} \leq n - \overline{D} - \overline{G} + \underline{G} - 2 \Longleftrightarrow 3 \cdot \underline{G} \leq n - \overline{D} - \underline{G} - 2 \tag{61}$$

$$\Longrightarrow \underline{G} \leq \left\lfloor \frac{n - \overline{D} - \underline{G} - 2}{3} \right\rfloor \tag{62}$$

$$\Longrightarrow \underline{G} \leq \left\lfloor \frac{n - \overline{D} - \overline{G} - 2}{3} \right\rfloor = \left\lfloor \frac{n - \overline{G} - \overline{D} - \min(\underline{D}, 1) - 1}{\min(\underline{D}, 1) + 2} \right\rfloor \tag{63}$$

$$\Longrightarrow \overline{G} \leq \left\lfloor \frac{n - \overline{G} - \overline{D} - \min(\underline{D}, 1) - 1}{\min(\underline{D}, 1) + 2} \right\rfloor + \underline{G} \tag{64}$$

□

4.4 Conjecture (65)

We prove the selected conjecture (65), which was found by the Bound Seeker.

$$\underline{D} \leq \begin{cases} 0 & \text{if } G \leq 1 \\ \left\lfloor \dfrac{n - \overline{G} + 1 - G}{G - 1} \right\rfloor & \text{if } G > 1 \end{cases} \tag{65}$$

Proof (Conjecture (65)). If $G \leq 1$ then there is no inter-distance of 0s between stretches of 1s. This means that $\underline{D} = 0$.

If $G > 1$ then there is a stretch of 1s that has a size equal to $\overline{G}$ and there are $G - 1$ stretches of 1s that have a size at least equal to 1. Let N_0 be the number of 0s, which are between stretches of 1s. So we have $N_0 \leq n - \overline{G} - (G - 1)$. Let d_i (with $i \in [1 : G - 1]$) be the number of 0s of the i-th inter-distance between stretches of 1s. We have

$$N_0 = \sum_{i=1}^{G-1} d_i \geq (G - 1) \cdot \underline{D} \implies \underline{D} \leq \frac{N_0}{G - 1} \leq \frac{n - \overline{G} - (G - 1)}{G - 1} \tag{66}$$

$$\Longrightarrow \underline{D} \leq \left\lfloor \frac{n - \overline{G} - (G - 1)}{G - 1} \right\rfloor \tag{67}$$

□

4.5 Conjecture (68)

We prove the selected conjecture (68), which was found by the Bound Seeker.

$$\overline{D} \leq [G \geq 2] \cdot (n - G \cdot \underline{G} - G + 2) \tag{68}$$

Proof (Conjecture (68)*).* If $G \leq 1$ then there is no inter-distance of 0s between stretches of 1s. This means that $\overline{D} = 0$.

If $G \geq 2$ then there are G stretches of 1s of lengths at least equal to $\underline{G}$. There is also an inter-distance of 0s of length $\overline{D}$ and there are $G - 2$ inter-distances of 0s of lengths at least equal to one. This means that $\overline{D} \leq n - G \cdot \underline{G} - (G - 2) = n - G \cdot \underline{G} - G + 2$. □

4.6 **Conjecture** (69)

We prove the selected conjecture (69), which was found by the Bound Seeker.

$$GS \geq \underline{G}^2 \cdot G \tag{69}$$

Proof (Conjecture (69)*).*

$$GS = \sum_{i=1}^{G} g_i^2 \geq \underline{G}^2 \cdot G \tag{70}$$

□

4.7 **Conjecture** (71)

We prove the selected conjecture (71), which was found by the Bound Seeker.

$$GS \geq \overline{G} \cdot (\overline{G} + 1) \cdot \min(G, 1) + \overline{G} + G \tag{71}$$

Proof (Conjecture (71)*).* If $G = 0$ then $\overline{G} = 0$, which leads to $GS = 0$; hence, the property holds.

If $G \geq 1$, then $\underline{G} \geq 1$.

$$\underline{G} \geq 1 \Longleftrightarrow \underline{G} - 1 \geq 0 \tag{72}$$

$$\Longleftrightarrow \overline{G} \cdot (\underline{G} - 1) \geq 0 \tag{73}$$

$$\Longleftrightarrow \overline{G} \cdot (\underline{G} - 1) + \underline{G} \geq 1 \tag{74}$$

$$\Longleftrightarrow -1 \geq -\overline{G} \cdot (\underline{G} - 1) - \underline{G} \tag{75}$$

Multipliying the inequality by 2, we have

$$-2 \geq -2 \cdot \overline{G} \cdot \underline{G} + 2 \cdot (\overline{G} - \underline{G}) \tag{76}$$

Adding $\overline{G}^2 + \underline{G}^2 + G$ on each part of the inequality, we obtain

$$\overline{G}^2 + \underline{G}^2 + G - 2 \geq \overline{G}^2 + \underline{G}^2 - 2 \cdot \overline{G} \cdot \underline{G} + 2 \cdot (\overline{G} - \underline{G}) + G \tag{77}$$

Replacing $\overline{G}^2 + \underline{G}^2 - 2 \cdot \overline{G} \cdot \underline{G}$ by $(\overline{G} - \underline{G})^2$, we obtain

$$\overline{G}^2 + \underline{G}^2 + G - 2 \geq (\overline{G} - \underline{G})^2 + 2 \cdot \overline{\underline{G}} + G \tag{78}$$

Replacing $\overline{G} - \underline{G}$ by $\overline{\underline{G}}$, we obtain

$$\overline{G}^2 + \underline{G}^2 + G - 2 \geq \overline{\underline{G}}^2 + 2 \cdot \overline{\underline{G}} + G = \overline{\underline{G}} \cdot (\overline{\underline{G}} + 1) + \overline{\underline{G}} + G \tag{79}$$

As $G \geq 1$, we have $\min(G, 1) = 1$. This leads to

$$\overline{G}^2 + \underline{G}^2 + G - 2 \geq \overline{\underline{G}} \cdot (\overline{\underline{G}} + 1) \cdot \min(G, 1) + \overline{\underline{G}} + G \tag{80}$$

Let g_i (with $i \in [1 : G]$) be the number of 1 in the i-th stretch of 1s. Note that with $g_{G-1} = \overline{G}$ and $g_G = \underline{G}$, we have $\forall i \in [1 : G - 2], g_i \geq 1$, which leads to

$$\sum_{i=1}^{G-2} g_i^2 \geq \sum_{i=1}^{G-2} 1 = G - 2 \tag{81}$$

Adding $\overline{G}^2 + \underline{G}^2$ on each part of the inequality, we obtain

$$\sum_{i=1}^{G} g_i^2 = \overline{G}^2 + \underline{G}^2 + \sum_{i=1}^{G-2} g_i^2 \geq \overline{G}^2 + \underline{G}^2 + G - 2 \tag{82}$$

$$\Longleftrightarrow GS = \sum_{i=1}^{G} g_i^2 = \overline{G}^2 + \underline{G}^2 + \sum_{i=1}^{G-2} g_i^2 \geq \overline{G}^2 + \underline{G}^2 + G - 2 \tag{83}$$

Finally, thanks to (80) and (83) we have

$$GS \geq \overline{G}^2 + \underline{G}^2 + G - 2 \geq \overline{\underline{G}} \cdot (\overline{\underline{G}} + 1) \cdot \min(G, 1) + \overline{\underline{G}} + G \tag{84}$$

□

4.8 Conjecture (85)

We prove the selected conjecture (85), which was found by the Bound Seeker.

$$GS \geq \max(\overline{G}^2 + 1 - [\underline{D} = 0] - [\overline{G} = 0], 0) \tag{85}$$

Proof (Conjecture (85)*).* If $\underline{D} = 0 \wedge \overline{G} = 0$, then $GS = 0 = \max(-1, 0) = \max(\overline{G}^2 + 1 - [\underline{D} = 0] - [\overline{G} = 0], 0)$.

If $\underline{D} = 0 \wedge \overline{G} \geq 1$, then there is just one stretch of 1s in the binary sequence. In that case $GS = \overline{G}^2 = \overline{G}^2 + 1 - [\underline{D} = 0] - [\overline{G} = 0] = \max(\overline{G}^2 + 1 - [\underline{D} = 0] - [\overline{G} = 0], 0)$.

If $\underline{D} \geq 1 \wedge \overline{G} \geq 1$, there are at least two stretches of 1s: one of length $\overline{G}$ and another of length at least 1. Therefore, $GS \geq \overline{G}^2 + 1 = \overline{G}^2 + 1 - [\underline{D} = 0] - [\overline{G} = 0] = \max(\overline{G}^2 + 1 - [\underline{D} = 0] - [\overline{G} = 0], 0)$. □

4.9 **Conjecture** (86)

We prove the selected conjecture (86), which was found by the Bound Seeker.

$$GS \leq \begin{cases} \max({N_1}^2 + G - 1, 0) & \text{if } G \leq 1 \\ \max((N_1 - G + 1)^2 + G - 1, 0) & \text{otherwise} \end{cases} \tag{86}$$

To prove this conjecture, we first prove the following Theorem 1.

Theorem 1 (maximisation of $S = \sum_i^P y_i^2$). *Let $y_1, y_2, \cdots, y_P$ be non-negative integers whose sum is equal to n and which maximise $S = \sum_{i=1}^P y_i^2$. Then the largest integer is equal to $y_1 = n - (P - 1)$ and the $P - 1$ remaining integers are all equal to* 1.

Proof. Let be the distribution of values among the integers y_i with $\forall i \in [1 : P]$ such that the maximum integer has the value $n - (P - 1)$ and the other integers are all equal to 1. So the sum of squares of the integers of this distribution is equal to $S_0 = (n - (P - 1))^2 + P - 1$. We will now show that for any other distribution of values among the integers y_i, we have $S \leq S_0$:

Any distribution other than the one that gives S_0 can be obtained by removing m occurrences of 1 from the largest value y_1 of the distribution for S_0, and then distributing these m values to the other initially equal 1 values. So we have

$\forall i \in [2:P], j_i \in \mathbb{N}$,

$$S = (y_1 - m)^2 + \sum_{i=2}^{P}(1+j_i)^2 \text{ with } \sum_{i=2}^{P} j_i = m < y_1 \tag{87}$$

$$S = y_1^2 + m^2 - 2 \cdot m \cdot y_1 + \sum_{i=2}^{P}(1 + j_i^2 + 2 \cdot j_i) \tag{88}$$

$$S = y_1^2 + m^2 - 2 \cdot m \cdot y_1 + \sum_{i=2}^{P} 1 + \sum_{i=2}^{P} j_i^2 + 2 \cdot \sum_{i=2}^{P} j_i \tag{89}$$

$$\text{As } \sum_{i=2}^{P} 1 = P - 1 \text{ and } \sum_{i=2}^{P} j_i = m, \text{ we have :}$$

$$S = y_1^2 + m^2 - 2 \cdot m \cdot y_1 + P - 1 + 2 \cdot m + \sum_{i=2}^{P} j_i^2 \tag{90}$$

$$\text{After rearranging each term, we obtain :}$$

$$S = y_1^2 + P - 1 + m^2 - 2 \cdot m \cdot y_1 + 2 \cdot m + \sum_{i=2}^{P} j_i^2 \tag{91}$$

$$\text{As } S_0 = y_1^2 + P - 1, \text{ we have :}$$

$$S = S_0 + \left(m^2 - 2 \cdot m \cdot y_1 + 2 \cdot m + \sum_{i=2}^{P} j_i^2\right) \tag{92}$$

We now show that the term in the parenthesis of (92) is negative. That means

$$m^2 - 2 \cdot m \cdot y_1 + 2 \cdot m + \sum_{i=2}^{P} j_i^2 \leq 0 \tag{93}$$

For that, we first express $2 \cdot m \cdot y_1$ with m^2 :

$$\begin{aligned} y_1 = y_1 - m + m &\iff 2 \cdot m \cdot y_1 = 2 \cdot m \cdot (y_1 - m + m) \\ &\iff 2 \cdot m \cdot y_1 = 2 \cdot m \cdot (y_1 - m) + 2 \cdot m^2 \end{aligned} \tag{94}$$

Then, according to (94), we replace $2 \cdot m \cdot y_1$ by $2 \cdot m \cdot (y_1 - m) + 2 \cdot m^2$ in the left term $m^2 - 2 \cdot m \cdot y_1 + 2 \cdot m + \sum_{i=2}^{P} j_i^2$ of the inequality (93). So we have

$$m^2 - 2 \cdot m \cdot y_1 + 2 \cdot m + \sum_{i=2}^{P} j_i^2 = 2 \cdot m - 2 \cdot m \cdot (y_1 - m) - m^2 + \sum_{i=2}^{P} j_i^2 \tag{95}$$

Now we show that the two parts $2 \cdot m - 2 \cdot m \cdot (y_1 - m)$ and $-m^2 + \sum_{i=2}^{P} j_i^2$ of (95) are negative:

– For the first term, we have

$$m < y_1 \Longrightarrow y_1 - m \geq 1 \tag{96}$$

$$\Longrightarrow 2 \cdot m \cdot (y_1 - m) \geq 2 \cdot m \tag{97}$$

$$\Longrightarrow 2 \cdot m - 2 \cdot m \cdot (y_1 - m) \leq 0 \tag{98}$$

– For the 2nd term, we have

$$m^2 = (\sum_{i=2}^{P} j_i) \cdot (\sum_{k=2}^{P} j_k) = \sum_{i=2}^{P} j_i(\sum_{k=2}^{P} j_k) = \sum_{i=2}^{P} j_i(j_i + \sum_{i \neq k}^{P} j_k) \tag{99}$$

$$m^2 = \sum_{i=2}^{P} j_i^2 + \sum_{i=2}^{P} \sum_{i \neq k}^{P} j_i \cdot j_k \tag{100}$$

$$\text{As } \sum_{i=2}^{P} \sum_{i \neq k}^{P} j_i \cdot j_k \geq 0 \text{ we have } m^2 \geq \sum_{i=2}^{P} j_i^2 \tag{101}$$

$$\text{which leads to } -m^2 + \sum_{i=2}^{P} j_i^2 \leq 0 \tag{102}$$

So thanks to $2 \cdot m - 2 \cdot m \cdot (y_1 - m) \leq 0$ and $-m^2 + \sum_{i=2}^{P} j_i^2 \leq 0$, we have the sum of the two previous terms, which give

$$2 \cdot m - 2 \cdot m \cdot (y_1 - m) - m^2 + \sum_{i=2}^{P} j_i^2 \leq 0 \tag{103}$$

And according to equality (95), we obtain the inequality (93) that we wanted to prove. Then, according to that inequality, we have

$$S_0 + \left(m^2 - 2 \cdot m \cdot y_1 + 2 \cdot m + \sum_{i=2}^{P} j_i^2 \right) \leq S_0 \tag{104}$$

So thanks to equality (92) we finally have

$$S \leq S_0 \tag{105}$$

□

Proof (Conjecture (86)*).* If $G = 0$, then $N_1 = 0$. This means that $GS = 0 = \max({N_1}^2 + G - 1, 0)$.

If $G = 1$, then the binary sequence has just one stretch of 1s. This means that $GS = {N_1}^2 = \max({N_1}^2 + G - 1, 0)$.

If $G > 1$, we have a binary sequence S_0' where the length of the largest stretch of 1s is $N_1 - (G - 1)$ and the length of the other $G - 1$ remaining stretches of 1s is one. So according to Theorem 1, we identify N_1 as n, G as P and the lengths of the stretches as the integers y_i, which lead to $GS \leq (N_1 - G + 1)^2 + G - 1$. And as $(N_1 - G + 1)^2 + G - 1 \geq 0$, we have

$$(N_1 - G + 1)^2 + G - 1 = \max((N_1 - G + 1)^2 + G - 1, 0) \tag{106}$$

This finally leads to $GS \leq \max((N_1 - G + 1)^2 + G - 1, 0)$. □

4.10 Conjecture (107)

We prove the selected conjecture (107), which was found by the Bound Seeker.

$$GS \leq \begin{cases} \max({N_1}^2, 0) & \text{if } \overline{D} = 0 \wedge \min(N_1, 1) = 1 \\ 0 & \text{if } \overline{D} = 0 \wedge \min(N_1, 1) = 0 \\ \max((N_1 - 2)^2 + 2, 0) & \text{if } \overline{D} \geq 1 \end{cases} \tag{107}$$

Proof (Conjecture (107)*).* If $\overline{D} = 0 \wedge \min(N_1, 1) = 1$, this means that $G \geq 1$. So the binary sequence has at least a stretch of 1s. In addition, as $\overline{D} = 0$, the binary sequence has inter-distances of 0s with the same length or no inter-distance of 0s. And when there is no inter-distance of 0s, we have $G = 1$ and $GS = {N_1}^2$. According to Theorem 1, by identifying $n = N_1$ and $G = P = 1$, the maximum value of the sum of squares of lengths of stretches of 1s that we can have in a binary sequence is $S_0 = (N_1 - (G-1))^2 + G - 1 = {N_1}^2$ where the number of 1s in the sequence is N_1. So that binary sequence has just one stretch of 1s of length N_1. This means that $GS = {N_1}^2 = \max({N_1}^2, 0)$.

If $\overline{D} = 0 \wedge \min(N_1, 1) = 1$, then $N_1 = 0$. So there are no stretches of 1s in the binary sequence. This means that $GS = 0$.

If $\overline{D} \geq 1$, then there are at least two inter-distances of 0s. This means that there are at least three stretches of 1s. According to Theorem 1, we have $GS \leq (N_1 - (G-1))^2 + G - 1$. Note that a binary sequence is a distribution of N_1 values of 1 between G stretches of 1s. So when the number G of stretches of 1s decreases, it increases the lengths of these stretches and therefore the sum of squares of lengths of stretches of 1s also increases. So $(N_1 - (G-1))^2 + G - 1$ reaches his maximum value $(N_1 - 2)^2 + 2$ when G reaches his minimum value 3 and we clearly find the expression of the conjecture. □

4.11 Conjecture (108)

We prove the selected conjecture (108), which was found by the Bound Seeker.

$$DS \geq \underline{D}^2 \cdot (G-1) \tag{108}$$

Proof (Conjecture (108)*).* If there is no inter-distance of 0s, we have $\underline{D} = 0$ and $DS = 0 = 0^2 \cdot (G-1) = \underline{D}^2 \cdot (G-1)$. So the conjecture holds.

Otherwise, if there is at least one inter-distance of 0s, let d_i (with $i \in [1 : G-1]$) be the number of 0s of the i-th inter-distance of 0s between stretches of 1s.

$$DS = \sum_{i=1}^{G-1} d_i^2 \text{ and } \forall i \in [1 : G-1], d_i \geq \underline{D} \tag{109}$$

$$\implies DS \geq \underline{D}^2 \cdot (G-1) \tag{110}$$

□

4.12 **Conjecture** (111)

We prove the selected conjecture (111), which was found by the Bound Seeker.

$$DS \geq \begin{cases} 0 & \text{if } G \leq 1 \\ \max((\underline{\overline{D}}+1)^2 + G - 2, 0) & \text{otherwise} \end{cases} \tag{111}$$

Proof (Conjecture (111)*).* If $G \leq 1$, then there is no inter-distance of 0s between stretches of 1s. So $DS = 0$.

If $G \geq 2$, then there is at least one inter-distance of 0s between stretches of 1s, and we distinguish two cases:

- When $G = 2$, there is just one inter-distance. So $\underline{D} = \overline{D}$ and $\underline{\overline{D}} = 0$. The smallest value of DS is the square of the smallest length of that inter-distance. And the smallest value of the inter-distance is 1. Then, we have

$$DS \geq 1^2 = 1 = \max(1,0) = \max((0+1)^2 + 2 - 2, 0) \tag{112}$$

$$\implies DS \geq \max((0+1)^2 + 2 - 2, 0) = \max((\underline{\overline{D}}+1)^2 + G - 2, 0) \tag{113}$$

$$\implies DS \geq \max((\underline{\overline{D}}+1)^2 + G - 2, 0) \tag{114}$$

  So the conjecture holds for the case $G = 2$.
- When $G > 2$, we have at least two inter-distances: the largest and smallest inter-distances with respective lengths $\overline{D}$ and $\underline{D}$. Note that $\overline{D} = \underline{\overline{D}} + \underline{D}$ and that there are $G - 1$ inter-distances in the binary sequence. So we have

$$DS = \overline{D}^2 + \underline{D}^2 + \sum_{i=1}^{G-3} d_i^2 = (\underline{\overline{D}} + \underline{D})^2 + \underline{D}^2 + \sum_{i=1}^{G-3} d_i^2 \tag{115}$$

  So, for given values of G and $\underline{\overline{D}}$, the sum of squares DS of lengths of inter-distances of 0s between stretches of 1s reaches its minimum value when the lengths of the inter-distances are minimum. That is $\forall i \in [1 : G-3], \underline{D} = d_i = 1$. So we have

$$DS \geq (\underline{\overline{D}}+1)^2 + 1 + \sum_{i=1}^{G-3} 1 = (\underline{\overline{D}}+1)^2 + G - 2 \tag{116}$$

  And because $(\underline{\overline{D}}+1)^2 + G - 2 \geq 0$, we have

$$DS \geq (\underline{\overline{D}}+1)^2 + G - 2 = \max((\underline{\overline{D}}+1)^2 + G - 2, 0) \tag{117}$$

  So finally we have

$$DS \geq \max((\underline{\overline{D}}+1)^2 + G - 2, 0) \tag{118}$$

□

4.13 **Conjecture** (119)

We prove the selected conjecture (119), which was found by the Bound Seeker.

$$DS \geq \overline{D}^2 \tag{119}$$

Proof (Conjecture (119)*).* If there is no inter-distance of 0s, we have $\underline{D} = 0$ and $DS = 0 = 0^2 = \underline{D}^2$. So the conjecture holds.

If there is at least one inter-distance of 0s, then we have

$$DS = \overline{D}^2 + \sum_{i=1}^{G-2} d_i^2 \tag{120}$$

As $\sum_{i=1}^{G-2} d_i^2 \geq 0$, we have $DS \geq \overline{D}^2$. □

4.14 **Conjecture** (121)

We prove the selected conjecture (121), which was found by the Bound Seeker.

$$DS \leq \begin{cases} 0 & \text{if } N_1 \leq 1 \\ (n - N_1)^2 & \text{otherwise} \end{cases} \tag{121}$$

Proof (Conjecture (121)*).* If $N_1 \leq 1$, then there is at most one stretch of 1s in the binary sequence. This means that there is no inter-distance of 0s between stretches of 1s. So $DS = 0$.

If $N_1 \geq 2$, then there are $n - N_1$ 0s to distribute among inter-distances of 0s between stretches of 1s. And, according to (102), the distribution that gives the maximum value of DS is when the binary sequence has just one inter-distance of 0s of length $n - N_1$ which is between two stretches of 1s. And because we have $N_1 \geq 2$, it is possible to build two stretches of 1s. So we can conclude that $DS \leq (n - N_1)^2$. □

4.15 **Conjecture** (122)

We prove the selected conjecture (122), which was found by the Bound Seeker.

$$DS \leq \begin{cases} \max((n - N_1 - (G-2))^2 + G - 2, 0) & \text{if } G \geq 2 \\ \max(G - 2, 0) & \text{otherwise} \end{cases} \tag{122}$$

Proof (Conjecture (122)*).* If $G \leq 1$, then there is no inter-distance of 0s between stretches of 1s in the binary sequence. So $DS = 0 = \max(G - 2, 0)$.

If $G \geq 2$, then there are $n - N_1$ values of 0 to distribute among $G - 1$ inter-distances of 0s between stretches of 1s in the binary sequence. According to Theorem 1, the distribution that gives the maximum value of DS is the one where the largest inter-distance of 0s has a length of $n - N_1 - (G - 2)$ and the remaining $G - 2$ inter-distances of 0s have a length of 1. Which means that $DS \leq (n - N_1 - (G - 2))^2 + G - 2$. □

4.16 Conjecture (123)

We prove the selected conjecture (123), which was found by the Bound Seeker.

$$\overline{G} \leq \begin{cases} n & \text{if } G = 1 \wedge \overline{D} = 0 \\ \min(G, 1) & \text{if } G \neq 1 \wedge \overline{D} = 0 \\ n - \overline{D} - (G-2) \cdot \underline{D} - G + \min(G, 1) & \text{if } G \neq 1 \wedge \overline{D} \geq 1 \end{cases} \tag{123}$$

Proof (Conjecture (123)*).* If $G = 1 \wedge \overline{D} = 0$, then there is no inter-distance of 0s between stretches of 1s in the binary sequence. So the maximum value of $\overline{G}$ is this case is n.

If $G \neq 1 \wedge \overline{D} = 0$, then $G = 0$. In this case $\overline{G} = 0 = \max(0, 1) = \max(G, 1)$.

If $G \neq 1 \wedge \overline{D} \geq 1$, then $G \geq 2$. Which means that $\min(G, 1) = 1$. It also means that there is a largest inter-distance of 0s of length $\overline{D}$, and $G-2$ remaining inter-distances of 0s of lengths equal, at least, to $\underline{D}$ which are all between stretches of 1s. Also there are $G-1$ stretches of 1s of lengths at least equal to 1, and the largest stretch of 1 of length $\overline{G}$. All this leads to

$$n = \overline{G} + \sum_{i=1}^{G-1} g_i + \overline{D} + \sum_{i=1}^{G-2} d_i \tag{124}$$

$$\text{As } g_i \geq 1 \text{ and } d_i \geq \underline{D}, \text{ we have } n \geq \overline{G} + (G-1) + \overline{D} + (G-2) \cdot \underline{D} \tag{125}$$

$$\text{So } \overline{G} \leq n - \overline{D} - (G-2) \cdot \underline{D} - (G-1) \tag{126}$$

$$\overline{G} \leq n - \overline{D} - (G-2) \cdot \underline{D} - G + 1 \tag{127}$$

$$\overline{G} \leq n - \overline{D} - (G-2) \cdot \underline{D} - G + \min(G, 1) \tag{128}$$

□

4.17 Conjecture (129)

We prove the selected conjecture (129), which was found by the Bound Seeker.

$$GS \leq \begin{cases} \max(n^2, 0) & \text{if } G = 1 \wedge \overline{D} = 0 \\ \max((\min(G, 1))^2 + G - 1, 0) & \text{if } G \neq 1 \wedge \overline{D} = 0 \\ \max((n - \overline{D} - (G-2) \cdot \underline{D} - G + 1)^2 + G - 1, 0) & \text{if } G \neq 1 \wedge \overline{D} \geq 1 \end{cases} \tag{129}$$

Proof (Conjecture (129)*).* To get the maximum value of GS, we need to get the maximum value of $\overline{G}$. The proof of Conjecture (123) gives the maximum value of $\overline{G}$, and according to Theorem 1, Conjecture (129) is proved. □

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